

Supersymmetry and non-Abelian geometric phase for a free particle on a circle with point-like interactions

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Abstract. Though not so widely appreciated in the literature, supersymmetric quantum mechanics provides an ideal playground for studying non-Abelian geometric phase, because supersymmetry always guarantees degeneracies in energy levels. In this paper we first present a simple supersymmetric model for a free particle on a circle with point-like interactions that exhibits $\mathcal{N} = 2$ supersymmetry and doubly degenerate energy levels. We then show that Berry's connection in this model is given by the Wu-Yang-like magnetic monopole in $SU(2)$ Yang-Mills gauge theory. This article is largely based on our recent work [1].

1. Introduction

Berry's geometric phase [2] and its non-Abelian generalization [3] are ubiquitous phenomena in quantum physics when model parameters are adiabatically driven along a closed loop on the parameter space. They appear in a wide spectrum of physics and have applications in many disciplines, ranging from condensed matter physics to nuclear and high energy physics.

The purpose of this paper is to introduce a new example of quantum mechanical models that exhibits nontrivial non-Abelian geometric phase. In order to realize non-Abelian geometric phase, we must at least have:

- (i) degenerate energy levels; and
- (ii) a certain number of (experimentally controllable) parameters.

Let us first look at the criterion (i). As is well-known, degeneracies in energy levels are closely tied to symmetries of the systems. A typical example of such symmetries is time-reversal invariance, which results in the Kramers degeneracy in Fermi systems and plays an important role in geometric phase and/or topological invariants in molecular systems [4, 5, 6]. In this paper we focus on yet another symmetry which induces spectral degeneracy: that is, *supersymmetry*, which also guarantees degenerate energy levels between “bosonic” and “fermionic” degrees of freedom of the system.¹

¹ This is always true except for the ground state(s): In supersymmetric quantum mechanics the ground state degeneracy is related to the topological index $\text{Tr}_{\mathcal{H}}(-1)^F$ rather than symmetries [7]. In [8, 9] non-Abelian geometric phase over the space of degenerate ground states has been studied in the context of supersymmetric quantum mechanics.

Let us next look at the criterion (ii). A typical example of (experimentally controllable) parameters discussed in the context of geometric phase is electromagnetic field. Here we would like to point out that generic point-like interactions may also serve the purpose, because point-like interactions are all known to be described by $U(2)$ family of boundary conditions [10] and characterized by $\dim U(2) = 4$ independent continuous parameters. Though, as far as we know, $U(2)$ family of point-like interactions have not yet realized in laboratory experiments, they are completely allowed in quantum mechanics and hence will provide a promising ingredient of future nanotechnology.

The aim of the paper is to present a canonical example of quantum mechanical models with point-like interactions that exhibits supersymmetry, spectral degeneracy, and nontrivial non-Abelian geometric phase.² As in any study of theoretical physics, it is tempting to construct a minimal model that captures all the essence of the ideas. In this paper we consider a single spinless particle on a circle of circumference 2ℓ that freely propagates in the bulk yet interacts only at the origin $x = 0$ and its antipodal point $x = \ell$, where $x = 0$ and $x = 2\ell$ are identified. The bulk dynamics is governed by the Schrödinger equation for a free particle

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = E\psi(x), \quad x \neq 0, \ell. \quad (1)$$

Point-like interactions, on the other hand, are all described by the boundary conditions consistent with *Kirchhoff's law* of probability current at $x = 0$ and ℓ [10]:

$$j(0_+) = j(2\ell_-) \quad \text{and} \quad j(\ell_-) = j(\ell_+), \quad (2)$$

where $j(x) = \frac{\hbar}{2im}(\psi^*(x)\psi'(x) - \psi'^*(x)\psi(x))$ is the local probability current (with prime (') indicating the derivative with respect to x) and $j(a_{\pm})$ stand for the limits approaching $x = a$ from above and from below. The conditions (2) are quadratic in wavefunctions, but they can be linearized and are known to enjoy the following $U(2)$ family of solutions at each point [10]:

$$(1_2 - U) \begin{pmatrix} \psi(0_+) \\ \psi(2\ell_-) \end{pmatrix} - iL_0(1_2 + U) \begin{pmatrix} \psi'(0_+) \\ -\psi'(2\ell_-) \end{pmatrix} = 0, \quad U \in U(2), \quad (3a)$$

$$(1_2 - \tilde{U}) \begin{pmatrix} \psi(\ell_-) \\ \psi(\ell_+) \end{pmatrix} + iL_0(1_2 + \tilde{U}) \begin{pmatrix} \psi'(\ell_-) \\ -\psi'(\ell_+) \end{pmatrix} = 0, \quad \tilde{U} \in U(2), \quad (3b)$$

where 1_2 stands for the 2×2 unit matrix and L_0 is an arbitrary length scale that needs to be introduced on account of dimensional analysis. Equations (3a) and (3b) describe the most general point-like interactions in quantum mechanics consistent with Kirchhoff's law of probability current (2), or, equivalently, self-adjointness of the free Hamiltonian. The full parameter space of the model is thus $U(2) \times U(2)$. Though energy levels are not degenerate in general, there indeed exists a subspace $\mathcal{M}_{\text{SUSY}} \subset U(2) \times U(2)$ in which $\mathcal{N} = 2$ supersymmetry³ emerges and energy levels become doubly degenerate (up to the question of ground state degeneracy). As shown in [1], the supersymmetric subspace $\mathcal{M}_{\text{SUSY}}$ is given by the following direct product:

$$\mathcal{M}_{\text{SUSY}} = U(1) \times \frac{U(2)}{U(1) \times U(1)} \cong S^1 \times S^2. \quad (4)$$

It can be shown that only the first factor controls the energy eigenvalues and the second factor does not affect the energy spectrum at all: the coset factor $U(2)/(U(1) \times U(1))$ describes the isospectral family of the system and hence provides an ideal playground for studying geometric

² In [11, 12] Abelian geometric phase has been studied in simple models of point-like interactions.

³ Here \mathcal{N} refers to the number of real (i.e. self-adjoint) supercharges.

phase. The goal of the paper is to report that the Berry connection on $\mathcal{M}_{\text{SUSY}}$ with fixed $U(1)$ parameter is given by a magnetic monopole-like configuration in non-Abelian gauge theory.

The organization of the paper is as follows. In section 2 we construct $\mathcal{N} = 2$ supersymmetric quantum mechanics in this system and solve the energy eigenvalue problems. We then compute non-Abelian Berry's connection on the parameter space $U(2)/(U(1) \times U(1)) \cong S^2$ in section 3 and show that it is nothing but the Wu-Yang-like magnetic monopole in $SU(2)$ Yang-Mills gauge theory. Section 4 is devoted to conclusions and discussions.

In the rest of the paper we will work in the units $\hbar = 2m = 1$.

2. Supersymmetric point-like interactions

The key to construct $\mathcal{N} = 2$ supersymmetric quantum mechanics for a free particle on a circle with two point-like interactions is to find the generic form of fermion parity operator $(-1)^F$, which should satisfy $((-1)^F)^2 = 1$ and have two eigenvalues $+1$ and -1 . In this section we first introduce $(-1)^F$ in our model and then discuss point-like interaction invariant under supersymmetry transformations. We also explicitly construct $\mathcal{N} = 2$ supersymmetry algebra and solve the superspectrum in this model.

2.1. Folding trick

In the following discussions it is convenient to work in the so-called *folding picture*,⁴ which is just a change of viewpoint from “scalar” quantum mechanics (i.e. quantum mechanics with scalar-valued wavefunction) on a circle of circumference 2ℓ to “vector” quantum mechanics (i.e. quantum mechanics with vector-valued wavefunction) on an interval of length ℓ . To do this, let us consider the wavefunction on the upper- and lower-semicircles separately and embed them into a single two-component vector-valued function as follows:

$$\Psi(x) := \begin{pmatrix} \psi(x) \\ \psi(2\ell - x) \end{pmatrix}, \quad 0 < x < \ell. \quad (5)$$

(Throughout the paper we will use boldface symbols for vectors.) The boundary conditions (3a) and (3b) are then cast into the following forms:

$$(1_2 - U)\Psi(0) - iL_0(1_2 + U)\Psi'(0) = 0, \quad (6a)$$

$$(1_2 - \tilde{U})\Psi(\ell) + iL_0(1_2 + \tilde{U})\Psi'(\ell) = 0. \quad (6b)$$

Here and hereafter we simply write $\Psi(0)$ and $\Psi(\ell)$ for $\Psi(0_+)$ and $\Psi(\ell_-)$. In the folding picture the Hilbert space is $\mathcal{H} = L^2(0, \ell) \otimes \mathbb{C}^2$ and the free Hamiltonian that acts on \mathcal{H} is given by the 2×2 matrix-valued operator, $H = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$, where $h = -d^2/dx^2$. The boundary conditions (6a) and (6b) specify the most general self-adjoint domain of the matrix-valued operator H .

Let us next introduce the fermion parity $(-1)^F$ in this model. As discussed in [15, 1], $(-1)^F$ is given by the following hermitian unitary operator:

$$\mathcal{Z} : \Psi(x) \mapsto (\mathcal{Z}\Psi)(x) := Z\Psi(x), \quad (7)$$

where Z is a generic 2×2 hermitian unitary matrix satisfying $Z = Z^\dagger = Z^{-1}$, or $Z^2 = 1_2$. Such hermitian unitary matrix is parameterized as $Z = \mathbf{n} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices and $\mathbf{n} = (n_1, n_2, n_3)$ is a real unit 3-vector fluffing the condition $n_1^2 + n_2^2 + n_3^2 = 1$. The parameter space of Z is therefore 2-sphere S^2 , which turns out to provide the coset factor in the supersymmetric parameter space (4). Notice that the unitary transformation (7) commutes with the Hamiltonian and leaves the Schrödinger equation unchanged. However, \mathcal{Z} does not

⁴ Folding trick has been developed in the context of boundary conform field theory [13, 14].

leave the boundary conditions (6a) and (6b) unchanged in general and hence is not a symmetry for generic point-like interactions. It is easy to show that the boundary conditions remain invariant under \mathcal{Z} if and only if both U and \tilde{U} commute with Z , $[U, Z] = [\tilde{U}, Z] = 0$ [16, 1]. The solutions to these conditions are parameterized as follows:

$$U = e^{i\alpha_+} P_+ + e^{i\alpha_-} P_-, \quad (8a)$$

$$\tilde{U} = e^{i\tilde{\alpha}_+} P_+ + e^{i\tilde{\alpha}_-} P_-, \quad (8b)$$

where $P_{\pm} = (1_2 \pm Z)/2$ are projection operators and $\{\alpha_+, \alpha_-, \tilde{\alpha}_+, \tilde{\alpha}_-\}$ are eigenphases of U and \tilde{U} . The fermion parity $(-1)^F = \mathcal{Z}$ becomes the symmetry of the system if and only if U and \tilde{U} are given by (8a) and (8b). In this case $(-1)^F = \mathcal{Z}$ provides a good grading operator such that the Hilbert space splits into two orthogonal subspaces, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_{\pm} = \{\Psi \in \mathcal{H} \mid \mathcal{Z}\Psi = \pm\Psi\}$ are “bosonic” and “fermionic” sectors of the model. Correspondingly, any element $\Psi \in \mathcal{H}$ of the Hilbert space \mathcal{H} is decomposed as follows:

$$\Psi(x) = \Psi_+(x) + \Psi_-(x), \quad (9)$$

where

$$\Psi_{\pm}(x) = P_{\pm}\Psi(x) = \Psi_{\pm}(x)e_{\pm}. \quad (10)$$

Here e_{\pm} are orthonormal eigenvectors of Z that satisfy the eigenvalue equations $Ze_{\pm} = \pm e_{\pm}$, the orthonormality $e_{\alpha}^{\dagger}e_{\beta} = \delta_{\alpha\beta}$ and the completeness $e_+e_+^{\dagger} + e_-e_-^{\dagger} = 1_2$. Notice that the projection operators can also be written as $P_{\pm} = e_{\pm}e_{\pm}^{\dagger}$. In what follows we will call the set of eigenvectors $\{e_+, e_-\}$ the basis and $\Psi_{\pm} = e_{\pm}^{\dagger}\Psi$ the components. It is easy to check that the \mathcal{Z} -invariant boundary conditions completely decouple into the “bosonic” and “fermionic” parts and the components Ψ_{\pm} satisfy the following Robin boundary conditions:

$$\sin\left(\frac{\alpha_{\pm}}{2}\right)\Psi_{\pm}(0) + L_0 \cos\left(\frac{\alpha_{\pm}}{2}\right)\Psi'_{\pm}(0) = 0, \quad (11a)$$

$$\sin\left(\frac{\tilde{\alpha}_{\pm}}{2}\right)\Psi_{\pm}(\ell) - L_0 \cos\left(\frac{\tilde{\alpha}_{\pm}}{2}\right)\Psi'_{\pm}(\ell) = 0. \quad (11b)$$

2.2. Supersymmetric boundary conditions

Let us next introduce supersymmetry transformations. To this end, let us first introduce the following one-parameter family of first-order differential operators:

$$A_{\alpha}^{\pm} = \pm \frac{d}{dx} + \frac{1}{L(\alpha)}, \quad (12)$$

where $L(\alpha) := L_0 \cot(\alpha/2)$. An important observation is that the free Hamiltonian $h = -d^2/dx^2$ is factorized as $h = A_{\alpha}^{\mp}A_{\alpha}^{\pm} - 1/L(\alpha)^2$ such that the Schrödinger equations $h\Psi_{\pm} = E\Psi_{\pm}$ can be written as $A_{\alpha}^{-}A_{\alpha}^{+}\Psi_+ = (E + 1/L(\alpha)^2)\Psi_+$ and $A_{\alpha}^{+}A_{\alpha}^{-}\Psi_- = (E + 1/L(\alpha)^2)\Psi_-$, which imply the following relations:

$$A_{\alpha}^{+}\Psi_+(x) = \sqrt{E + \frac{1}{L(\alpha)^2}}\Psi_-(x), \quad (13a)$$

$$A_{\alpha}^{-}\Psi_-(x) = \sqrt{E + \frac{1}{L(\alpha)^2}}\Psi_+(x). \quad (13b)$$

Table 1. Supersymmetric boundary conditions.

	“bosonic” sector	“fermionic” sector
type DD	$\Psi_+(0) = 0 = \Psi_+(\ell)$	$(A_\alpha^- \Psi_-)(0) = 0 = (A_\alpha^- \Psi_-)(\ell)$
type RR	$(A_\alpha^+ \Psi_+)(0) = 0 = (A_\alpha^+ \Psi_+)(\ell)$	$\Psi_-(0) = 0 = \Psi_-(\ell)$
type DR	$\Psi_+(0) = 0 = (A_\alpha^+ \Psi_+)(\ell)$	$(A_\alpha^- \Psi_-)(0) = 0 = \Psi_-(\ell)$
type RD	$(A_\alpha^+ \Psi_+)(0) = 0 = \Psi_+(\ell)$	$\Psi_-(0) = 0 = (A_\alpha^- \Psi_-)(\ell)$

These equations give the supersymmetry transformations that map the “bosonic” state Ψ_+ to the “fermionic” state Ψ_- and vice versa. It should be emphasized that the Robin boundary conditions (11a) and (11b) are not invariant under these transformations for generic values of α_\pm and $\tilde{\alpha}_\pm$. The boundary conditions invariant under the supersymmetry transformations are classified in [1], and it is shown that there are four distinct supersymmetric boundary conditions, which are summarized in table 1. We note that the supersymmetric boundary conditions admit only one parameter α , which is an angle parameter and provides the $U(1)$ factor in (4).

2.3. $\mathcal{N} = 2$ supersymmetry algebra

Now we are in a position to introduce $\mathcal{N} = 2$ supersymmetry algebra that consists of a single self-adjoint Hamiltonian H , two self-adjoint supercharges Q_1 and Q_2 , and a single fermion parity $(-1)^F$. For the following discussions it is convenient to consider the nilpotent supercharges Q^\pm which are given by the linear combinations $Q^\pm = (Q_1 \pm iQ_2)/2$. Let us first work in the basis in which the fermion parity becomes diagonal. In that basis the wavefunction becomes $\Psi = (\Psi_+, \Psi_-)^T$ such that the operators H , $(-1)^F$ and Q^\pm are given by the following standard forms:

$$H = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad (14a)$$

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (14b)$$

$$Q^+ = \begin{pmatrix} 0 & 0 \\ A_\alpha^+ & 0 \end{pmatrix}, \quad (14c)$$

$$Q^- = \begin{pmatrix} 0 & A_\alpha^- \\ 0 & 0 \end{pmatrix}. \quad (14d)$$

These operators satisfy the $\mathcal{N} = 2$ supersymmetry algebra with the trivial central extension

$$((-1)^F)^2 = 1, \quad (15a)$$

$$(Q^\pm)^2 = 0, \quad (15b)$$

$$Q^\pm (-1)^F = -(-1)^F Q^\pm, \quad (15c)$$

$$Q^+ Q^- + Q^- Q^+ = H + c, \quad (15d)$$

where $c = 1/L(\alpha)^2$ is the trivial center that commutes with all the operators. In a generic basis, these operators that act on the Hilbert space $\mathcal{H} = L^2(0, \ell) \otimes \mathbb{C}^2$ are given by $H = h \otimes 1_2$, $(-1)^F = 1 \otimes Z$, $Q^\pm = A_\alpha^\pm \otimes XP_\pm$, where X is a generic 2×2 hermitian unitary matrix that satisfies the anticommutation relation $XZ = -ZX$ and $XP_\pm X = P_\mp$.

It should be noted that, under the supersymmetric boundary conditions, Q^+ and Q^- are indeed adjoint to each other such that the combination $H + c$ becomes positive semidefinite. In other words, the spectrum of the Hamiltonian $\text{Spec}(H)$ is bounded from below, $\text{Spec}(H) \geq -c = -1/L(\alpha)^2$, and the lower bound is saturated by the zero-modes of Q^\pm .

2.4. $\mathcal{N} = 2$ superspectrum

In order to compute Berry's connection explicitly we have to know the spectrum and normalized energy eigenfunctions $\{\Psi_{\pm,n} = \Psi_{\pm,n} \mathbf{e}_\pm\}_{n=0}^\infty$. It is a straightforward exercise to solve the Schrödinger equations $\hbar \Psi_{\pm,n} = E_n \Psi_{\pm,n}$ for the components Ψ_\pm with the supersymmetric boundary conditions in table 1. Here are the solutions:

- (i) *Type DD*. In this case the positive energy levels are all doubly degenerate and normalized energy eigenfunctions are found to be of the forms

$$\Psi_{+,n}(x) = \sqrt{\frac{2}{\ell}} \sin(\sqrt{E_n}x), \quad (16a)$$

$$\Psi_{-,n}(x) = \sqrt{\frac{2}{\ell} \frac{1}{1 + E_n L(\alpha)^2}} \left[\sin(\sqrt{E_n}x) + \sqrt{E_n} L(\alpha) \cos(\sqrt{E_n}x) \right], \quad (16b)$$

where $E_n = (n\pi/\ell)^2$, $n = 1, 2, 3, \dots$. In addition to these positive energy eigenstates, there exists a single negative energy eigenstate at $E = E_0 = -1/L(\alpha)^2$ in the “fermionic” sector, which is the ground state of the model and takes the following form:

$$\Psi_{-,0}(x) = \sqrt{\frac{2}{L(\alpha)} \frac{1}{e^{2\ell/L(\alpha)} - 1}} \exp\left(\frac{x}{L(\alpha)}\right). \quad (17)$$

- (ii) *Type RR*. As in the previous case the positive energy levels are doubly degenerate too and normalized energy eigenfunctions take the following forms:

$$\Psi_{+,n}(x) = \sqrt{\frac{2}{\ell} \frac{1}{1 + E_n L(\alpha)^2}} \left[\sin(\sqrt{E_n}x) - \sqrt{E_n} L(\alpha) \cos(\sqrt{E_n}x) \right], \quad (18a)$$

$$\Psi_{-,n}(x) = \sqrt{\frac{2}{\ell}} \sin(\sqrt{E_n}x), \quad (18b)$$

where $E_n = (n\pi/\ell)^2$, $n = 1, 2, 3, \dots$. In addition, there is a single negative energy eigenstate at $E = E_0 = -1/L(\alpha)^2$ in the “bosonic” sector, whose eigenfunction is given by

$$\Psi_{+,0}(x) = \sqrt{\frac{2}{L(\alpha)} \frac{1}{1 - e^{-2\ell/L(\alpha)}}} \exp\left(-\frac{x}{L(\alpha)}\right). \quad (19)$$

- (iii) *Type DR*. In this case all the energy levels are doubly degenerate and normalized energy eigenfunctions take the following forms:

$$\Psi_{+,n}(x) = \sqrt{\frac{2}{\ell + \frac{L(\alpha)}{1 + E_n L(\alpha)^2}}} \sin(\sqrt{E_n}x), \quad (20a)$$

$$\Psi_{-,n}(x) = \sqrt{\frac{2}{\ell + \frac{L(\alpha)}{1 + E_n L(\alpha)^2}}} \sin(\sqrt{E_n}(\ell - x)), \quad (20b)$$

where E_n ($E_0 < E_1 < \dots$) are given by the real roots of the transcendental equation

$$\tan(\sqrt{E}\ell) = -\sqrt{E}L(\alpha). \quad (21)$$

We note that the ground state energy E_0 becomes negative when $-\ell < L(\alpha) < 0$.

(iv) *Type RD.* As in the previous case all the energy levels are doubly degenerate too and normalized energy eigenfunctions take the following forms:

$$\Psi_{+,n}(x) = \sqrt{\frac{2}{\ell - \frac{L(\alpha)}{1+E_n L(\alpha)^2}}} \sin(\sqrt{E_n}(\ell - x)), \quad (22a)$$

$$\Psi_{-,n}(x) = \sqrt{\frac{2}{\ell - \frac{L(\alpha)}{1+E_n L(\alpha)^2}}} \sin(\sqrt{E_n}x), \quad (22b)$$

where E_n ($E_0 < E_1 < \dots$) are the real roots of the transcendental equation

$$\tan(\sqrt{E}\ell) = \sqrt{E}L(\alpha). \quad (23)$$

3. Non-Abelian Berry's connection: The Wu-Yang-like monopole

Let us finally study geometric phase in $\mathcal{N} = 2$ supersymmetric quantum mechanics in this system, whose parameter space is $\mathcal{M}_{\text{SUSY}} = U(1) \times U(2)/(U(1) \times U(1))$. To this end, let us consider a simple time-dependent situation in which the $U(1) \cong S^1$ parameter is kept fixed yet the coset $U(2)/(U(1) \times U(1)) \cong S^2$ parameters are adiabatically driven along a closed loop γ on S^2 . Then, if the initial state is an element of the subspace $\mathcal{H}_n = \text{span}\{\Psi_{+,n}, \Psi_{-,n}\}$, the final state remains in the subspace \mathcal{H}_n and coincides with the initial states up to the time-dependent trivial dynamical phase $\exp(-iE_n T)$ and the time-independent nontrivial non-Abelian geometric phase $W_\gamma(A)$ given by the formula [3]:

$$W_\gamma(A) = \mathcal{P} \exp \left(i \oint_\gamma A \right), \quad (24)$$

where $A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$ is the hermitian matrix-valued 1-form on the parameter space $U(2)/(U(1) \times U(1)) \cong S^2$ defined by

$$A_{\alpha\beta} = i \int_0^\ell dx \Psi_{\alpha,n}^\dagger(x) d\Psi_{\beta,n}(x), \quad \alpha, \beta \in \{+, -\}. \quad (25)$$

Here d stands for the exterior derivative on S^2 . Substituting the solutions one readily finds that the Berry connection takes the following form:

$$A = \begin{pmatrix} ie_+^\dagger de_+ & iK_n(\alpha) e_+^\dagger de_- \\ iK_n(\alpha) e_-^\dagger de_+ & ie_-^\dagger de_- \end{pmatrix}, \quad (26)$$

where $K_n(\alpha)$ is the overlap integral between the components

$$K_n(\alpha) = \int_0^\ell dx \Psi_{+,n}(x) \Psi_{-,n}(x). \quad (27)$$

Under the local gauge transformation $\Psi_{\alpha,n} \mapsto \tilde{\Psi}_{\alpha,n} = \Psi_{\beta,n} g_{\beta\alpha}$, where $g = (g_{\beta\alpha})$ is a 2×2 unitary matrix that depends on the S^2 parameters, the Berry connection transforms as follows:

$$A \mapsto \tilde{A} = g^\dagger A g + i g^\dagger d g = g^\dagger A g - i (d g^\dagger) g. \quad (28)$$

For the following discussions it is convenient to work in the gauge given by

$$g = \begin{pmatrix} e_+^\dagger \\ e_-^\dagger \end{pmatrix}. \quad (29)$$

In this gauge the Berry connection takes the simple form $\tilde{A} = \frac{i}{2}(1 - K_n(\alpha))Z dZ$ [1]. Let us now parameterize the hermitian unitary matrix Z as $Z = \frac{\mathbf{r}}{r} \cdot \boldsymbol{\sigma}$, where $\mathbf{r} = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. A straightforward calculation gives $\tilde{A} = \tilde{A}_i dx_i$, where

$$\tilde{A}_i = \epsilon_{ijk} \frac{x_j}{r^2} \frac{\sigma_k}{2} (1 - K_n(\alpha)). \quad (30)$$

Notice that, when $K_n(\alpha) = 0$, this is nothing but the Wu-Yang magnetic monopole which is a classical solution of pure $SU(2)$ Yang-Mills gauge theory. (For the Wu-Yang monopole, see e.g. [17] for review.) Note also that, as shown in [1], (30) is gauge equivalent to the non-Abelian Berry connection discussed in the context of geometric phase in the adiabatic decoupling limit of diatomic molecule [18]. (See also [19, 20].)

4. Conclusions and discussions

In this paper we discussed non-Abelian geometric phase in $\mathcal{N} = 2$ supersymmetric quantum mechanics with point-like interactions. We computed Berry's connection on the parameter space of supersymmetric point-like interactions and showed that it is given by the Wu-Yang-like magnetic monopole in $SU(2)$ Yang-Mills gauge theory. It would be interesting to point out here that, if K_n was a function of r and had the asymptotic behaviors $K_n(r) \rightarrow 1$ as $r \rightarrow 0$ and $K_n(r) \rightarrow 0$ as $r \rightarrow \infty$, the Berry connection would be the celebrated singularity-free 't Hooft-Polyakov monopole [21, 22] in $SU(2)$ Yang-Mills-Higgs theory. It would be also quite interesting to point out that, if the zero modes of Q^+ and Q^- coexisted in the spectrum, the overlap integral $K_0(\alpha) = \int_0^\ell dx \Psi_{+,0}(x) \Psi_{-,0}(x)$ between the components (17) and (19) would become

$$K_0(\alpha) = \frac{\ell/L(\alpha)}{\sinh(\ell/L(\alpha))}. \quad (31)$$

The Berry connection (30) would then take the form

$$\tilde{A}_i = \epsilon_{ijk} \frac{x_j}{r^2} \frac{\sigma_k}{2} \left(1 - \frac{\ell/L(\alpha)}{\sinh(\ell/L(\alpha))} \right), \quad (32)$$

which exactly coincides with the Bogomolny-Prasad-Sommerfield (BPS) monopole [23, 24] in $SU(2)$ Yang-Mills-Higgs theory under the identification $\ell/L(\alpha) \equiv evr$, where e is the electric charge and v is the vacuum expectation value of Higgs field. As in the model studied in [9], it might be interesting to explore a model of supersymmetric point-like interactions in which the “bosonic” and “fermionic” zero-modes coexist and the Berry connection is given by the 't Hooft-Polyakov monopole that saturates the BPS bound.

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